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# LANG-BOMBIERI'S CONJECTURES

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## Contents



Abstract. In this article we generalize Iitaka-Viehweg conjectures for geometric varieties([Km4]) to those for arithmetic ones. They are applicable to Diophantine problems as well as geometiric Diophantine problems.

## 1. Introduction

We shall prove the following conjectures proposed by Lang and Bombieri([SL]):

**Conjecture 1.** *Let $K$ be an arithmetic field and $X$ a variety defined over $K$. Assume that $X$ be a variety of general type. Then it has no dense set of $K$-rational points in $X$.*

This is well known as Mordell's conjecture in case of curves of genus $\geq 2$ and is shown by Faltings([F]).

There is an analogue of the conjecture, which is proposed by Noguchi([No], [Km1], [Km2]. [Km3]):

**Conjecture 2.** *Let $X$ and $S$ be algebraic varieties over the field of the complex numbers. Assume that $X/S$ be a fibre space with the geometric generic fibre of general type. If $X/S$ has a dense set of rational sections in $X$, then* $\mathrm{var}(X/S) = 0$.

First applying Kollar-Kawamata's theorem([Kaw], [Ko2], [Ko1], [Mat]), we shall prove the conjecture 2. Secondary we shall show an arithmetic version the conjecture 1 later.

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## 2. Review Iitaka-Viehweg conjecture

Iitaka proposed the following conjecture in 1970:

**Conjecture 3.** *([I]) Let $X/S$ be a fibre space over the field of the complex numbers and $X_{\bar{\eta}}$ the geometric generic fibre of $X/S$. Then $\kappa(X) \geq \kappa(X_{\bar{\eta}}) + \kappa(S)$.*

Viehweg raised the following

**Conjecture 4.** *([V]) Let $f : X \to S$ be a fibre space $X/S$ with the geometric generic fibre of Kodaira dimension $\geq 0$. Then there exists a number $m$ such that*

$$\kappa(\det f_* \omega_{X/S}^{\otimes m}) \geq \mathrm{var}(X/S).$$

**Notation 1.** *([SGA], [GG]. [I], [V]. [Ko2], [Dix], [MiyPet]), [Mat])*

(1) *Let $k$ be a field. A geometrically irreducible, reduced, smooth scheme $X$ over $k$ is said to be a non singular variety over $k$.*
(2) *Let $X$ be a non singular variety of dimension $d$ and $\Omega_X^1$ the differential sheaf over $X$. Let $\omega_X$ denote $\Omega_X^d$.*
(3) *A connected proper surjective morphism $f : X \to S$ of non singular varieties $X$ and $S$ is said to be a fibre space $X/S$.*
(4) *Let $f : X \to S$ be a fibre space $X/S$. $\omega_{X/S}$ denotes $\omega_X \otimes f^*\omega_S^{-1}$.*
(5) *Let $L$ be an invertible sheaf over $X$. $\kappa(L)$ denotes the maximal dimension of the image variety of the rational map $X \to \mathbf{P}(\Gamma(X, L^{\otimes m}))$ defined by $\Gamma(X, L^{\otimes m}) \otimes \mathrm{O}_X \to L^{\otimes m}$ for $m >> 0$. We call $\kappa(L)$ Iitaka dimension of $L$.*
(6) *$\kappa(\omega_X)$ is said to be Kodaira dimension of $X$, which is denoted by $\kappa(X)$.*
(7) *Let $X/S$ be a fibre space. We denote by $\mathrm{var}(X/S)$ the minimal dimension of $T$ such that there exists a generically finite morphism $S' \to S$ in which $X \times_S S'$ is birationally equivalent to $S' \times_T X_0$ for some varieties $T$, $X_0$ with $X_0/T$ a fibre space. This dimension is called Viehweg dimension of a fibre space $X/S$.*

$$\begin{tikzcd}
 & X \times S' \arrow[r] \arrow[dl] \arrow[dd] & X_0 \times_T S' \arrow[ddl] \arrow[dddr] & \\
X \arrow[d] & & & \\
S & S' \arrow[l] \arrow[ddr] & & \\
 & & & X_0 \arrow[dl] \\
 & & T &
\end{tikzcd}$$

(8) *The category of bands(*[Gir], [SGA]*) of profinite groups(*[Se], [Shatz], [RBZL], [Zuo]*) is defined in the following. The objects are the profinite groups and the arrows are the homomorphisms of profinite groups modulo inner automorphisms.*

(9) *A* $\mathbf{Q}$*-divisor* $D$ *is said to be effective if* $\kappa(D) \geq 0$.

We have proved the two conjecture 3 and 4. Recall the sketch of the proof.

(1) Let $X/S$ be a fibre space with the geometric generic fibre of Kodaira dimension $\geq 0$([I]). We may assume $\mathrm{var}(X/S) = \dim S$ by Viehweg's comment. Construct a generically finite cover $Y$ of $X$ such that
   (a) $Y$ is a variety of general type or a variety with the abundant canonical divisor,
   (b) $\max_{m>0} \kappa(\det f_* \omega_{X/S}^{\otimes m}) = \max_{m>0} \kappa(\det g_* \omega_{Y/S}^{\otimes m})$, where $\max_{m>0} \kappa(\det g_* \omega_{Y/S}^{\otimes m}) \geq \mathrm{var}(Y/S)$ is obtained by Kollar and Kawamata independently.

(2)
$$\mathrm{var}(Y/S) \geq \mathrm{var}(X/S).$$

The following Mochizuki's theorem is available to prove (2) above.

**Theorem 1.** *(*[Mch]*) Let* $p$ *be a prime number. Let* $K$ *be a subfield of a finitely generated field extension of* $\mathbb{Q}_p$*. Let* $L, M$ *be function fields of arbitrary dimension over* $K$*. Let* $\mathrm{Hom_{Spec(K)}(Spec(L), Spec(M))}$ *be the set of* $K$*-morphisms from* $\mathrm{Spec}(L)$ *to* $\mathrm{Spec}(M)$*. Let* $\mathrm{Hom}^{open}_{\Gamma_K}(\Gamma_L, \Gamma_M)$ *over* $\Gamma_K$*, the set of open continuous homomorphisms of profinite groups considered up to composition with an inner automorphism arising from* $\mathrm{Ker}(\Gamma_M, \Gamma_K)$*, where* $\Gamma_L$ *and* $\Gamma_M$ *are the absolute Galois groups of* $L$ *and* $M$*, respectively. Then the natural map* $\mathrm{Hom}_{\mathrm{Spec}(K)}(\mathrm{Spec(L), Spec(M)}) \to \mathrm{Hom^{open}_{\Gamma_K}(\Gamma_L, \Gamma_M)}$ *is bijective.*

Let $p$ be a prime number. Let $K$ be a subfield of a finitely generated field extension of $\mathbb{Q}_p$. Note that there exists an isomorphism $\iota : \bar{K} \cong \mathbf{C}$.

Let $X_{\bar{\eta}}$ be the geometric generic fibre of $X/S$. Then there exists a variety $F_{K_0}$ and a finite extension field $K_0$ of $\mathbf{Q}$ such that $F_{K_0} \times_{K_0} \bar{K}_0 \cong X_{\bar{\eta}}$.

Let $\mathrm{Bir}_{\mathbf{C}}(X_{\bar{\eta}}) = \mathrm{Bir}_{\bar{K}_0}(F_{K_0} \otimes_{K_0} \bar{K}_0)$. There exists a definition field $K$ of $\mathrm{Bir}_{\bar{K}_0}(F_{K_0} \otimes_{K_0} \bar{K}_0)$ such that $\mathrm{Bir}_K(F_K) \otimes_K \bar{K} \cong \mathrm{Bir}_{\mathbf{C}} X_{\bar{\eta}}$ and that $K$ is a finite extension field of $\mathbf{Q}$ since $\mathrm{Bir}(X_{\bar{\eta}})$ is an algebraic group. Fix $K$ once for all.

Let $\pi : \Gamma_{F_K} \to \Gamma_K$ denote the structure map associated to $F_K \to \mathrm{Spec}(K)$, which is a surjection since $K$ is algebraically closed in the rational function field of $F$ . Let $Z(\Gamma_{F_K})$ denote the centre of $\Gamma_{F_K}$. Then $\pi$ induces $\pi : Z(\Gamma_{F_K}) \to Z(\Gamma_K)$.

Hence consider a crossed module

$$(\pi^{-1}(Z(\Gamma_K)) \to \mathrm{Aut}_{\Gamma_K}(F_K)).$$

There exists an exact sequence by Breen([Breen1], [Breen2], [BJ])

$$0 \to Z(\Gamma_{F_K})[1] \to (\pi^{-1}(Z(\Gamma_K)) \to \mathrm{Aut}_{\Gamma_K}(F_K)) \to \mathrm{Out}_{\Gamma_K}(F_K) \to 1$$

and a long exact sequence:

$$0 \to H^1(\Gamma_{S_K}, Z(\Gamma_{F_K})[1]) \to H^1(\Gamma_{S_K}, (\pi^{-1}(Z(\Gamma_K)) \to \mathrm{Aut}_{\Gamma_K}(F_K))) \to H^1(\Gamma_{S_K}, \mathrm{Out}_{\Gamma_K}(F_K))$$

We proved the following lemma making use of revised Matsumura's theorem.

**Lemma 1.** *([Mat]) Let $F_K$ be a variety of $\kappa(F_K) \geq 0$. Then $\mathrm{Bir}_K(F_K)$ has at most countable connected components and the connected component which contains an identity element is of finite type over $K$.*

We can take base change $K'/K$ which is a finite extension so that the image of the extension element $\Gamma_{X_K}$ associated to $X/S$ is a trivial map in $H^1(\Gamma_{S_K}, \mathrm{Out}_{\Gamma_K}(F_K)) \cong \mathrm{Hom}(\Gamma_{S_K}, \mathrm{Out}_{\Gamma_K}(F_K))$. Thus $\Gamma_{X_K}$ is going up to $H^1(\Gamma_{S_K}, Z(\Gamma_{F_K})[1])$, which turns out to a central extension.

To prove $\mathrm{var}(Y/S) \geq \mathrm{var}(X/S)$, we have proved that it is enough to show that $\mathrm{var}(Y/S) = 0$ implies $\mathrm{var}(X/S) = 0$.

Assume $\mathrm{var}(Y/S) = 0$. Then the extension $1 \to \Gamma_{F_{\bar{K}}} \to \Gamma_{X_K} \to \Gamma_{S_K} \to 1$ associated to a fibre space $X/S/\mathrm{Spec}\,K$ splits. Hence $\Gamma_{X_K}$ is a semi-direct product. This extension is central since the argument above. On the other hand, a central extension which is a semi-direct product is a trivial product.

By Breen's theory([Breen1], [Breen2], [AM]), we have

$$H^1(\Gamma_{S_K}, (\pi^{-1}(Z(\Gamma_K)) \to \mathrm{Aut}_{\Gamma_K}(F_K))) \cong \mathrm{Ext}_{\Gamma_K}(\Gamma_{S_K}, \Gamma_{F_K}),$$

whose neutral element is $\Gamma_{S_K} \times_{\Gamma_K} \Gamma_{F_K}$. The sketch of the proof finishes.

## 3. Application to Geometric Diophantus problems

**Theorem 2.** *Let $X/S$ be a fibre space with the geometric generic fibre of general type. Assume that there is a dense set of rational sections of $X/S$ in $X$. Then $\mathrm{var}(X/S) = 0$.*

*Proof.* It suffices to prove the assertion for every curve $C$ on $S$. Namely we will prove for a fibre space $X_C/C$. Change the base curve $C$ by a normal curve $C'$ of genus $g(C') \geq 2$. Let $X/C$ denote such a fibre space and an image of a rational section $C_\lambda$, which is in fact a section of $X/C$.

Since $\kappa(\omega_{X/C}) \geq \kappa(\omega_{X_{\bar{\eta}}})$, there exist a number $m$ and the following commutative square for $\dim X = n$:

$$\begin{tikzcd}
\Omega_X^{\otimes mn} & \\
\mathcal{O}_X(mK_X) \arrow[u] & \\
\Omega_C^{\otimes m} \arrow[u] \arrow[r] & \Omega_C^{\otimes mn} \arrow[uul]
\end{tikzcd}$$

Restricting the commutative diagram above to a curve $C_\lambda$ on $X$, we have

$$\begin{tikzcd}
\Omega_X^{\otimes mn}|_{C_\lambda} \arrow[ddr] & \\
\mathcal{O}_X(mK_X)|_{C_\lambda} \arrow[u] & \\
\Omega_C^{\otimes m}|_{C_\lambda} \arrow[u] \arrow[r] & \Omega_C^{\otimes mn}|_{C_\lambda}
\end{tikzcd}$$

Thus we have the upper bound of the intersections

$$(C_\lambda, K_X) \leq n \deg \Omega_C^1 = n(2g(C) - 2).$$

Since $\kappa(C) = 1$ and $\kappa(\omega_X) \geq \kappa(\omega_{X_{\bar{\eta}}}) + \kappa(\omega_C)$, we obtain $\kappa(\omega_X) = \dim X$. Hence $\omega_X$ is big.

For any ample invertible sheaf $L$ there exists a number $\ell$ such that $L \hookrightarrow \omega_X^\ell$ (Kodaira's Lemma). We have the estimation of intersections $(C_\lambda, L) \leq \ell(C_\lambda, \omega_X) \leq n\ell(2\mathrm{g}(C) - 2)$ for every curve which is not contained in the stable base locus of $\omega_X$. These curves are parametrized by Hilbert schemes $Hilb_X^{p(m)}$, where the Hilbert polynomials $p(m)$ are a finite number of polynomials since $(C_\lambda, L) \leq n\ell(2g(C) - 2)$. Hence one can find a variety $T$ which is a component of parametrizing Hilbert schemes and a dominant rational map $\psi$ over $C$ such that $\psi : T \times C \to X$ over $C$. See the following diagram:

$$\begin{tikzcd}
T & \mathbf{\Gamma} = \{C_\lambda\} \subset T \times X \arrow[l] \arrow[r] \arrow[dl] \arrow[dr] & X \arrow[d] \\
T \times C \arrow[u] \arrow[rr] & & C
\end{tikzcd}$$

Here $\mathbf{\Gamma}$ is the restriction of the universal family over $Hilb_X^{p(m)}$ to $T$. Hence $\mathbf{\Gamma}$ is birationally equivalent to $T \times C$. Therefore

$$0 = \mathrm{var}(T \times C/C) \geq \mathrm{var}(X/C)$$

and $\mathrm{var}(X/S) = 0$. $\square$

**Theorem 3.** *Let $k$ be an algebraically closed field of characteristic $0$. Let $X$ be a variety of general type over $k$ and $g$ a number which is not less than 1. Then the set of non singular curves $\{C_\lambda\}$ of genus $g$ in $X$ which do not contained in a fixed hyperplane forms a bounded set, i.e., given an ample invertible sheaf $L$ over $X$, $\chi(C_\lambda, L^{\otimes m})$ are a finite number of polynomials in $m$.*

*Proof.* Let $X \to \mathbf{P}^1$ be a fibre space obtained by Lefshetz pencil up to birational equivalence. For each curve $C_\lambda$ in $X$, we have a fibre space $X \times_{\mathbf{P}^1} C_\lambda \to C_\lambda$. Applying the argument above to these fibre spaces, we complete the proof. $\square$

Note that even if $g = 0$, the assertion holds since it is enough to change the direction of a homomorphism in the diagram above to $\omega_{C_\lambda}^{\otimes nm} \to \omega_{C_\lambda}$.

Next we shall prove the following result which is an analogue of Hermite's theorem.

**Theorem 4.** *Let $C$ be a curve, $S$ a set of points on $C$ and $d$ a number. Given a covering degree $\leq d$ of covers over $C$, there exists only a finite number of finite covers over $C$ which are etale outside $S$.*

*Proof.* Let $B$ be such a finite cover over $C$. There are a finite number of $\omega_{B/C}$ for all such covers $B$. Since $B \cong \mathbf{Proj}(\oplus_{m\geq 0}\omega_{B/C}^{\otimes m})$, we get the theorem. $\square$

**Theorem 5.** *Let $X$ be a variety and $C$, $D$ curves. Let $f : X \to C \times D$ be a proper surjective morphism. Assume $f$ is smooth over $C^o \times D^o$, where $C^o$ and $D^o$ are open subsets of $C$ and $D$, respectively. Then* $\mathrm{var}(f) \leq 1$.

To prove an analogue of Lang-Trotter's conjectures it seems to be applicable .

## 4. Arithmetic Iitaka-Viehweg conjecture

**Definition 1.** *Let $K$ be a finite extension of $\mathbf{Q}$. Let $\mathcal{O}_K$ denote the integral closure of $\mathbf{Z} \subset K$. We call an arithmetic variety if it is an irreducible and reduced scheme which is flat and projective over* $\operatorname{Spec}\mathcal{O}_K$*:*

$$f : \mathcal{X} \to \operatorname{Spec}\mathcal{O}_K$$

*and if, moreover, the base change $f : X \to \operatorname{Spec} K$ is an irreducible and reduced morphism.*

We propose an arithmetic Iitaka-Viehweg conjecture:

**Conjecture 5.** *Let $f : \mathcal{X} \to \mathcal{S}$ be an arithmetic fibre space defined over* $\operatorname{Spec}\mathcal{O}_K$ *and let the geometric generic fibre of the fibre space of Kodaira dimension $\geq 0$.*

(1) *If* $\dim\mathcal{S} \geq 2$, $\max_{m>0}\kappa(\det f_*\omega_{\mathcal{X}/\mathcal{S}}^{\otimes m}) \geq \mathrm{var}(\mathcal{X}/\mathcal{S}) \geq 1$.

(2) *If, especially,* $\mathcal{S} = \operatorname{Spec}\mathcal{O}_K$, $\max_{m>0}\kappa(\det f_*\omega_{\mathcal{X}/\mathcal{S}}^{\otimes m}) \geq \mathrm{var}(\mathcal{X}/\mathcal{S}) \geq 1$

Note that the latter conjecture is essential. By taking pull-back through $\operatorname{Spec} K \to \operatorname{Spec} \mathcal{O}_K$, we have an ordinary fibre space $X/S$ over $\operatorname{Spec} K$. By Iitaka's lemma, we obtain $\kappa(\det f_* \omega_{\mathcal{X}/\mathcal{S}}^{\otimes m}) \leq \kappa(\det f_* \omega_{X/S}^{\otimes m}) + 1$. Note that $\operatorname{var}(\mathcal{X}/\mathcal{S}) \geq \operatorname{var}(X/S) + 1$. If the geometric generic fibre is of general type, we can show $\max_{m>0} \kappa(\det f_* \omega_{\mathcal{X}/\mathcal{S}}^{\otimes m}) \geq \max_{m>0} \kappa(\det f_* \omega_{X/S}^{\otimes m}) + 1$. In general, construct a cover such that the fibre is of general type and that the same value as $\max_{m>0} \kappa(\det f_* \omega_{\mathcal{X}/\mathcal{S}}^{\otimes m})$ and we get $\max_{m>0} \kappa(\det f_* \omega_{\mathcal{X}/\mathcal{S}}^{\otimes m}) \geq \max_{m>0} \kappa(\det f_* \omega_{X/S}^{\otimes m}) + 1$.

**Lemma 2** (Moriwaki([Mo])). *Let $K$ be an algebraic field, $X$ a projective irreducible reduced variety over $K$ and $L$ an invertible sheaf over $X$. Then there exists a couple $(\mathcal{X}, \mathcal{L})$ such that*

(1) *$\mathcal{X} \to Spec(\mathcal{O}_K)$ is an arithmetic projective variety and its generic fibre is $X$.*
(2) *Forgetting the underling metric structure $\mathcal{L}$, it is $\mathbf{Q}$-equivalent to $L$.*

**Lemma 3.** *$f_* \omega_{\mathcal{X}/\mathcal{S}}^{\otimes m}$ is weakly positive for $m > 0$.*

*Proof.* Let $\mathcal{L}$ be an ample invertible sheaf over $\mathcal{S}$. Then $H^1(\mathcal{X}, \omega_{\mathcal{X}} \otimes f^*\mathcal{L}) \to H^1(\mathcal{X}, \omega_{\mathcal{X}} \otimes f^*\mathcal{L}^{\otimes m})$ is injective for $m \geq 2$ since Koll'ar-Viehweg results. Hence $H^1(\mathcal{S}, f_*\omega_{\mathcal{X}} \otimes \mathcal{L}) = 0$. Using the dualizing sheaf, we have

$$H^0(\mathcal{S}, \omega_{\mathcal{O}_K} \otimes \underline{Hom}_{\mathcal{O}_K}(f_*\omega_{\mathcal{X}} \otimes \mathcal{L}, \mathcal{O}_K)) = 0$$

Hence $f_*\omega_{\mathcal{X}/\mathcal{S}}$ is weakly positive. By Viehweg's method([V], [Fuj], [Ws]), we can complete the proof. □

**Lemma 4.** *Assume $\kappa(\omega_X) = \dim X$ and $\dim \mathcal{S} = 1$. Then $\deg \det f_* \omega_{\mathcal{X}/\mathcal{S}}^{\otimes m} > 0$.*

*Proof.* By weak positivity, $\deg \det f_* \omega_{\mathcal{X}/\mathcal{S}}^{\otimes m} \geq 0$. If $\deg \det f_* \omega_{\mathcal{X}/\mathcal{S}}^{\otimes m} = 0$, then $\operatorname{var}(\mathcal{X}/\mathcal{S} = 0$, which is absurd. □

**Proposition 1.** (1) *Assume $\kappa(\omega_X) = \dim X$ and $\dim \mathcal{S} = 1$. Then $\max_{m>0} \kappa(\det f_* \omega_{\mathcal{X}/\mathcal{S}}^{\otimes m}) \geq \operatorname{var}(\mathcal{X}/\mathcal{S}) \geq 1$.*
(2) *Assume $\kappa(\omega_X) = \dim X$. Then $\max_{m>0} \kappa(\det f_* \omega_{\mathcal{X}/\mathcal{S}}^{\otimes m}) \geq \operatorname{var}(\mathcal{X}/\mathcal{S}) \geq 1$.*

*Proof.* Take a generically finite cover $\mathcal{Y}$ such that

(1) $\max_{m>0} \kappa(\det f_* \omega_{\mathcal{Y}/\mathcal{S}}^{\otimes m}) = \max_{m>0} \kappa(\det f_* \omega_{\mathcal{X}/\mathcal{S}}^{\otimes m})$.
(2) $\kappa(Y) = \dim Y$, $\max_{m>0} \kappa(\det f_* \omega_{\mathcal{Y}/\mathcal{S}}^{\otimes m}) \geq \operatorname{var}(\mathcal{Y}/\mathcal{S}) \geq 1$

Making use of $\operatorname{var}(\mathcal{Y}/\mathcal{S}) \geq \operatorname{var}(\mathcal{X}/\mathcal{S})$, we obtain the theorem. □

## 5. Application of Arithmetic Iitaka-Viehweg conjecture

**Theorem 6.** *Let $X/S$ be a fibre space with $X$ of general type. Then $X$ has no dense set of rational points.*

In other words, we have the following theorem.

**Theorem 7.** *Let $\mathcal{X}/\mathcal{S}$ be an arithmetic fibre space with the generic fibre $X$ of general type. Then there exists no dense set of rational sections.*

*Proof.* We will apply the following theorem.

**Lemma 5** (Northcott([Szp]), [Mo])**.** *For any $\epsilon > 0$ and $M$ the set $\{x \in \mathbf{P}^n(\bar{\mathbf{Q}}|[\mathbf{Q}(x) : \mathbf{Q}] \le e, h_{nv}(x) \le M\}$ is finite.*

By Iitaka's conjecture, $\kappa(\omega_{\mathcal{X}/\mathcal{S}}) \ge 0$. Take the base change such that $[K : \mathbf{Q}] > 1$. We have

$$0 \to \mathcal{O}_K \to \omega_{\mathcal{O}_K} \to \Omega_{\mathcal{O}_K/\mathbf{Z}} \to 0.$$

$$\begin{tikzcd}
\Omega_{\mathcal{X}}^{\otimes mn} & \\
\mathcal{O}_X(mK_X) \arrow[u] & \\
\omega_{\mathcal{S}}^{\otimes m} \arrow[u] \arrow[r] & \Omega_{\mathcal{S}/\mathbf{Z}}^{\otimes mn} \arrow[uul]
\end{tikzcd}$$

Restrict the commutative diagram above to a curve $\mathcal{S}_\lambda$ on $X$, we have

$$\begin{tikzcd}
\Omega_{\mathcal{X}}^{\otimes mn}|_{\mathcal{S}_\lambda} \arrow[ddr] & \\
\omega_{\mathcal{X}}^{\otimes m}|_{\mathcal{S}_\lambda} \arrow[u] & \\
\omega_{\mathcal{S}}^{\otimes m}|_{\mathcal{S}_\lambda} \arrow[u] \arrow[r] & \Omega_{\mathcal{S}_\lambda/\mathbf{Z}}^{\otimes mn}
\end{tikzcd}$$

**Proposition 2.** *We have the following results:*

(1) *Let $K_\ell$ be the kernel such that the following becomes exact sequences :* $0 \to K_\ell \to \omega_{\mathcal{O}_K}^{\otimes n\ell} \to \Omega_{\mathcal{O}_K/\mathbf{Z}}^{\otimes n\ell} \to 0.$

(2) *We have an injection $\mathcal{O}_K \to K_\ell$ and*

(3)
$$H^1(\mathcal{S}_\lambda, \underline{Hom}(\omega_{\mathcal{X}/\mathbf{Z}}|_{\mathcal{S}_\lambda}, K_\ell)) \cong H^0(\mathcal{S}_\lambda, \omega_{\mathcal{O}_K} \otimes \omega_{\mathcal{X}/\mathbf{Z}}|_{\mathcal{S}_\lambda} \otimes K_\ell^{-1}) = 0$$
*for $\ell >> 0$.*

Let $\Phi_K$ denote the set of the places of $K$ at infinities and $r_1$ the number of the real places, $r_2$ the number of the complex places $\sigma$ such that $\sigma$ or $\bar{\sigma}$ belongs to $\Phi_K$([Szp]).

Note that $\deg \omega_{\mathcal{O}_K} = -2\xi(\mathcal{O}_K) = -2\log(2^{r_2}D^{-1/2})$([Szp]). Apply the following lemma to get the proof.

**Lemma 6.** *Let* $\cap_n \mathrm{Supp}(\mathrm{Coker}(H^0(C, nL) \otimes \mathcal{O}_C \to nL)) = \mathrm{SBs}(L)$. *Then there exists a number* $N$ *such that for any point* $x \in (X \setminus \mathrm{SBs}(L))(\bar{K})$, $h_{(\mathcal{C},\mathcal{L})}(x) \geq N$.

**Lemma 7** (Northcott([Szp])). *Let* $X$ *be a projective variety over* $\mathrm{Spec}\, K$ *and* $L$ *an ample invertible sheaf over* $X$. *Let* $h_L : X(\bar{K}) \to \mathbf{R}$ *denote the height function associated to* $L$. *Then for any positive real numbers* $\epsilon$, $M$, *the set* $\{x \in X(\bar{K}) | [K(x) : \mathbf{Q}] \leq \epsilon, h_L(x) \leq M\}$ *is a finite set.*

This completes the theorem. $\square$

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